Exam Statistical Physics

Friday, November 28, 2014

The total number of points is 50. Do not forget to write your name and the student number on the first sheet. Some useful constants and integrals can be found on the last page. Good luck.

1. **Ideal classical gas**

   Consider an ideal classical gas of N atoms at a temperature T. The gas occupies the volume V. The atom mass is m.

   (a) Show that the single-atom partition function, $z = \sum_{\mathbf{p}} e^{-\beta \varepsilon_{\mathbf{p}}}$, is given by

   $$z = \frac{V}{\lambda^3},$$

   where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is the termal de Broglie wave length. [5 points]

   (b) Show that the entropy of the gas is given by

   $$S = Nk_B \left[\ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2} \right].$$

   [5 points]

   Hint: Find the free energy of the gas F.

2. **Canonical ensemble**

   (a) Show that the average energy $\langle E \rangle$ of a system in contact with the heat bath can be calculated by differentiating its canonical partition function Z with respect to $\beta = 1/k_B T$

   $$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z$$

   [3 points]

   (b) Show that the average energy fluctuation is given by

   $$\langle (E - \langle E \rangle)^2 \rangle = \frac{\partial^2}{\partial \beta^2} \ln Z = -\frac{\partial \langle E \rangle}{\partial \beta}$$

   [4 points]

(c) Find the fluctuation of the energy of 1 mole of an ideal gas at a temperature T [3 points]

3. **Entropy of ideal gas**

   Two vessels contain each N atoms of Ar gas. Initially, the two gases are isolated from each other, the gases being at the same temperature T but at different pressures, P_1 and P_2. The partition separating the gases is removed. Find the change in entropy of the system when equilibrium has been re-established, in terms of the initial pressures P_1 and P_2. [10 points]

   Hint: Use equation of state and expressions for energy and entropy of ideal classical monoatomic gas. Derive them if necessary.

4. **Specific heat of gases**

   The table below shows the constant volume specific heat per molecule, c_V, of three different gases measured at 1 atm and 15 °C.

gas	c_V/k_B
Ar	1.50
NO	2.51
H_2S	3.06

   Can you make sense out of these numbers? [10 points]

   Hint: What can one conclude about the rotational and vibrational temperatures of these gases?

5. **Pressure of Fermi gas**

   Derive the pressure $P = P(n)$ of a free electron gas with the density $n = \frac{N}{V}$ electrons per unite volume, at the absolute zero of temperature. [10 points]

Constants, conversion factors and integrals:

$\hbar = 1.1 \times 10^{-34}$ J s $= 6.6 \times 10^{-16}$ eV s, $c = 3 \times 10^{8}$ m s^{-1}, $\hbar c = 2 \times 10^{-5}$ eV cm,

$m_e = 9.1 \times 10^{-31}$ kg, $m_e c^2 = 5.1 \times 10^5$ eV, $k_B = \frac{1\text{ eV}}{11606\text{ K}} = 1.38 \times 10^{-23}$ J K^{-1},

$N_A = 6.02 \times 10^{23}$, $R = k_B N_A = 1.99$ cal mole^{-1} K^{-1} $= 8.31$ J mole^{-1}K^{-1},

1 eV $= 1.6 \times 10^{-19}$ J, 1 cal $= 4.18$ J, 1 atm $= 760$ mmHg $= 1.01 \times 10^5$ N m^{-2},

$a_B = \frac{\hbar^2}{m_e e^2} = 0.53$ Å, 1Å $= 10^{-8}$ cm, Ry $= \frac{1}{2}\frac{\hbar^2}{m_e a_B^2} = \frac{1}{2}\frac{m_e e^4}{\hbar^2} = 13.6$ eV.

$$\int_0^\infty dx x^n e^{-x} = n!, \quad \int_0^\infty dx x^{2n+1} e^{-x^2} = \frac{n!}{2}, \quad \int_{-\infty}^\infty dx x^{2n} e^{-x^2} = \sqrt{\pi}\,\frac{(2n)!}{2^{2n} n!}, \quad n = 0, 1, 2, \ldots,$$

$$I_n = \int_0^\infty \frac{dx x^{n-1}}{e^x - 1} = (n-1)!\zeta(n), \quad n = 1, 2, \ldots, \quad I_3 \approx 2.4, \ I_4 = \frac{\pi^4}{15} \approx 6.5.$$